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On Bernstein Polynomials for Compact Lie Groups*

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We give the Bernstein polynomials for basic matrix entries of irreducible unitary representations of compact Lie group $SU(2)$. We also give an application to the analytic continuation of certain distributions on $SU(2)$, and finally we briefly describe the Bernstein polynomial for $B_{-} \times B$-semi-invariant functions on a semisimple complex Lie group.

1. INTRODUCTION

Let G be a compact connected Lie group. Let $A(G)$ be the ring of representative functions on G, that is, the ring generated by all matrix coefficients of all finite-dimensional representations of G. Let $f \in A(G)$, $f \neq 0$, and take $A_f = \{gf^{-\nu}: g \in A(G),\ \nu \geq 0\}$ the ring of rational functions whose denominators are nonnegative powers of f.

Let us take an independent variable s and consider the ring of polynomials $A_f[s]$. Although f^s has no sense as a function in G, let $A_f[s] \cdot f^s$ be the $A_f[s]$ free module generated by f^s. We shall extend formally the action of $\mathfrak{g} = \mathrm{Lie}(G)$ on f^s using elements of $A_f[s]$, as follows. Let $X \in \mathfrak{g}$; we define

$$X(f^s) = \frac{s}{f} X(f) \cdot f^s. \tag{1.1}$$

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By the product rule, we extend this action to $A_f[s] \cdot f^s$. Let $\mathscr{D}(G)$ be the ring of differential operators generated by $A(G)$ and $\mathfrak{g}$. Let us observe that the action defined in (1.1) can be extended to $\mathscr{D}(G)$. Allowing s to act multiplicatively, we have an action of $\mathscr{D}(G)[s]$.

By the theory of $\mathscr{D}$-modules, we have the following functional equation:

$$D(g,s)(f(g))^{s+1} = b(s)(f(g))^{s}, \tag{1.2}$$

where $D(g,s)$ belongs to the ring of differential operators $\mathscr{D}(G)[s]$ and $b(s) \in \mathbb{C}[s]$. The existence of such a functional equation has been proved explicitly by Björk (see [4]).

The polynomial $b(s)$ is not unique. There exists a monic polynomial of minimal degree that satisfies (1.2). Such polynomial is called the *Bernstein polynomial* attached to f. These polynomials, also known as b-functions, have been introduced by Bernstein [2, 3] as part of the solution of Gelfand's problem on the meromorphic of continuation of certain analytic distributions. For a detailed study of this facet of Bernstein's work, we refer to [4]. Kashiwara [5] shows that the roots of the Bernstein polynomial are negative rational numbers.

The explicit calculation of Bernstein polynomials has been the goal of many recent works; see, for example, [1], [7].

Since the computation of Bernstein polynomials is intimately related to the construction of the differential operator that satisfies (1.2), we construct such a differential operator (Theorem 2.1) when G is the compact Lie group $SU(2)$. Furthermore, we show that the polynomial that appears in this functional equation is a scalar multiple of the corresponding Bernstein polynomial (Theorem 3.1). More precisely, let $g \in SU(2)$; i.e.,

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $|\alpha|^2 + |\beta|^2 = 1$. Let us take the irreducible unitary representation of $SU(2)$ with highest weight l; then a basic matrix coefficient has the following expression:

$$t^l_{m,n}(g) = \frac{(-1)^{l-n}}{(l-m)!}\alpha^{-m-n}\beta^{-m+n}\frac{d^{(l-m)}}{dt^{(l-m)}}\bigg|_{t=|\alpha|^2}\left[t^{l+n}(1-t)^{l-n}\right], \tag{1.3}$$

where $(d^{(l-m)}/dt^{(l-m)})[t^{l+n}(1-t)^{l-n}]$ is related, by the Rodrigues' formula, with the well-known Jacobi polynomials (cf. [9, 10]).

For each pair (m,n), using the properties of Jacobi polynomials, we shall construct a differential operator $D(s)$ with coefficients in $A(SU(2))[s]$, such that

$$D(s)\big(t^l_{m,n}(g)\big)^{s+1} = p(s)\big(t^l_{m,n}(g)\big)^{s},$$

where $p(s) = \prod_j((s+1) - j/c_{m,n,l})$ ($c_{m,n,l}$ constants) is the Bernstein polynomial.

In Section 4, we give an application to analytic continuation of distributions, and in the last section we describe the Bernstein polynomials for $B_- \times B$-semi-invariant functions on a semisimple algebraic Lie group.

2. CONSTRUCTION OF A DIFFERENTIAL OPERATOR

Let $G = SU(2)$. Recall that $A(G) = \mathbb{C}[\alpha, \overline{\alpha}, \beta, \overline{\beta}]$ is the ring of representative functions of G. Let $t^l_{m,n}(g)$ as in (1.3) and fix m, n, and l. Let us define

$$p_\alpha(s) = \begin{cases} \left(\prod_{j=0}^{|m+n|-1} ((|m+n|)(s+1) - j)\right) & \text{if } m+n \neq 0 \\ 1 & \text{if } m+n = 0 \end{cases} \tag{2.1}$$

and

$$p_\beta(s) = \begin{cases} \left(\prod_{j=0}^{|m-n|-1} ((|m-n|)(s+1) - j)\right) & \text{if } m-n \neq 0 \\ 1 & \text{if } m-n = 0. \end{cases} \tag{2.2}$$

In this section we shall prove the following theorem. This theorem is an independent and constructive proof of the existence of the functional equation (1.2) in this particular case.

THEOREM 2.1. *Let $t^l_{m,n}(g)$ be the matrix coefficient of the irreducible unitary representation of the group $SU(2)$ with highest weight l. Then there exists a differential operator $D(s)$ with coefficients in $A(G)[s]$ such that*

$$D(s)\big(t^l_{m,n}(g)\big)^{s+1} = p(s)\big(t^l_{m,n}(g)\big)^{s}, \tag{2.3}$$

where $p(s) \in \mathbb{C}[s]$ is the least common multiple of p_α, p_β and $s+1$.

Remark. In the next section we shall show that $p(s)$ is a scalar multiple of the Bernstein polynomial.

To prove Theorem 2.1, we shall need the following lemma, which gives us a description of matrix entries:

LEMMA 2.2.

(1) *If* $m + n \geq 0$ *and* $m - n \geq 0$, *then*

$$t^l_{m,n}(g) = \overline{\alpha}^{(m+n)}\overline{\beta}^{(m-n)}P^l_{m,n}(|\alpha|^2).$$

(2) *If* $m + n < 0$ *and* $m - n \geq 0$, *then*

$$t^l_{m,n}(g) = \alpha^{-(m+n)}\overline{\beta}^{(m-n)}Q^l_{m,n}(|\alpha|^2).$$

(3) *If* $m + n \geq 0$ *and* $m - n < 0$, *then*

$$t^l_{m,n}(g) = \overline{\alpha}^{(m+n)}\beta^{-(m-n)}R^l_{m,n}(|\alpha|^2).$$

(4) *If* $m + n < 0$ *and* $m - n < 0$, *then*

$$t^l_{m,n}(g) = \alpha^{-(m+n)}\beta^{-(m-n)}S^l_{m,n}(|\alpha|^2),$$

where $P^l_{m,n}(t), Q^l_{m,n}(t), R^l_{m,n}(t), S^l_{m,n}(t)$ *are polynomials such that neither* 0 *nor* 1 *is their root, and the roots of these polynomials in* $(0, 1)$ *are simple. Furthermore,* $P^l_{m,n}(t) = C_{m,n,l}P^{(m+n,m-n)}_{l-m}(1 - 2t)$, *where* $C_{m,n,l}$ *is a constant, and* $P^{(m+n,m-n)}_{l-m}(t)$ *is a Jacobi polynomial.*

Proof. By the Rodrigues' formula we have

$$\frac{d^{(l-m)}}{dt^{(l-m)}}\left[t^{l+n}(1-t)^{l-n}\right] = t^{m+n}(1-t)^{m-n}P^{(m+n,m-n)}_{l-m}(1-2t),$$

where $P^{(m+n,m-n)}_{l-m}(t)$ is a Jacobi polynomial. This equation is taken as the definition of Jacobi polynomials, and it differs from the usual one (cf. [8, p. 67]) by a constant. Replacing in (1.3) and evaluating in $|\alpha|^2$ (recall that $|\alpha|^2 + |\beta|^2 = 1$), we have the following general formula:

$$t^l_{m,n}(g) = C_{m,n,l}\overline{\alpha}^{(m+n)}\overline{\beta}^{(m-n)}P^{(m+n,m-n)}_{l-m}(1-2|\alpha|^2). \tag{2.4}$$

In general, given a Jacobi polynomial $P^{(a,b)}_n(t)$ with $n \geq 0$ and $a, b \in \mathbb{Z}$, it has the following properties (cf. [8, p. 144]):

(i) If $a > 0$ and $b > 0$, neither 1 nor -1 is a root of $P^{(a,b)}_n(t)$.

(ii) If $a < 0$, 1 is a root of $P^{(a,b)}_n(t)$ with multiplicity $-a$.

(iii) If $b < 0$, -1 is a root of $P^{(a,b)}_n(t)$ with multiplicity $-b$.

(iv) All of the roots of $P^{(a,b)}_n(t)$ in $(-1, 1)$ are simple.

Define $P^l_{m,n}(|\alpha|^2) = C_{m,n,l} P_{l-m}^{(m+n,\,m-n)}(1 - 2|\alpha|^2)$. Then, by (2.4),(i) and (iv), we deduce statement (1) of this lemma.

Let us note that if $m + n < 0$ and $m - n \geq 0$, by (ii) we have

$$P_{l-m}^{(m+n,\,m-n)}\big(1 - 2|\alpha|^2\big) = \alpha^{-(m+n)} \tilde{Q}^l_{m,n}\big(|\alpha|^2\big),$$

where $\tilde{Q}^l_{m,n}$ is a polynomial with simple roots in $(0,1)$, and neither 0 nor 1 is a root of $\tilde{Q}^l_{m,n}$. Thus, defining $Q^l_{m,n} = C_{m,n,l}\tilde{Q}^l_{m,n}$ we have the second statement of the lemma.

Analogously, the remaining statements follow. ∎

Let us fix E_-, E_0, and E_+ a basis of $\mathfrak{su}(2)$ such that

$$[E_0, E_+] = E_+ \quad [E_0, E_-] = -E_- \quad [E_+, E_-] = 2E_0.$$

We shall study the action this basis on $t^l_{m,n}(g)$:

LEMMA 2.3. *Let $k \in \mathbb{N}$, and (π_l, V^l) is the irreducible unitary representation of $SU(2)$ with highest weight l; then*

(i) $E^k_- t^l_{m,n}(g) = \prod_{j=0}^{k-1}(l + n - j) t^l_{m,n-k}(g)$.

(ii) $E^k_+ t^l_{m,n}(g) = \prod_{j=0}^{k-1}(l - n - j) t^l_{m,n+k}(g)$.

(iii) $E^k_0 t^l_{m,n}(g) = \prod_{j=0}^{k-1}(n - j) t^l_{m,n}(g)$.

Proof. We shall prove (i). The other cases follow in the same way. By [10, Theorem 37.2] we have

$$\begin{aligned} E_- t^l_{m,n}(g) &= \frac{d}{dt}\bigg|_{t=0} t^l_{m,n}\big(g \exp(tE_-)\big) \\ &= \frac{d}{dt}\bigg|_{t=0} \big\langle \pi_l\big(g \exp(tE_-)\big) v_n, v_m\big\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \big\langle \pi_l\big(\exp(tE_-)\big) v_n, \pi_l\big(g^{-1}\big) v_m\big\rangle \\ &= \big\langle (E_- v_n), \pi_l\big(g^{-1}\big) v_m\big\rangle \\ &= (l + n)\big\langle \pi_l(g) v_{n-1}, v_m\big\rangle \end{aligned}$$

Hence, by induction, we complete the proof of this lemma. ∎

EXAMPLE 2.4. The smallest nontrivial example that we can consider is $l = 1/2$. In this case,

$$t^{1/2}(g) = \begin{pmatrix} t^l_{-1/2,-1/2}(g) & t^l_{-1/2,1/2}(g) \\ t^l_{1/2,-1/2}(g) & t^l_{1/2,1/2}(g) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Then by Lemma 2.3 we have

$$\begin{array}{lll} E_-(\alpha) = 0 & E_0(\alpha) = -\frac{1}{2}\alpha & E_+(\alpha) = \beta \\ E_-(\beta) = \alpha & E_0(\beta) = \frac{1}{2}\beta & E_+(\beta) = 0 \\ E_-(\overline{\alpha}) = -\overline{\beta} & E_0(\overline{\alpha}) = \frac{1}{2}\overline{\alpha} & E_+(\overline{\alpha}) = 0 \\ E_-\left(-\overline{\beta}\right) = 0 & E_0\left(-\overline{\beta}\right) = \frac{1}{2}\overline{\beta} & E_+\left(-\overline{\beta}\right) = \overline{\alpha} \end{array} \tag{2.5}$$

Remark. Observe that E_- acts as $-\overline{\beta}\alpha\, d/dt$ and E_+ as $\overline{\alpha}\beta\, d/dt$.

Fix m, n, and l. Let us prove Theorem 2.1 for a very simple case. This will be useful in understanding the ideas of the general proof.

Case $m+n=0$ and $m+n=0$. In this particular case, $t^l_{0,0}(g) = P_l^{(0,0)}(1-2|\alpha|^2)$, where $P_l^{(0,0)}(t)$ is a Legendre polynomial. It follows that $t(1-t)(d/dt)(P_l^{(0,0)}(1-2t))$ and $P_l^{(0,0)}(1-2t)$ are coprime, i.e., there exist polynomials $q_1(t)$ and $q_2(t)$ such that

$$q_1(t)P_l^{(0,0)}(1-2t) + q_2(t)t(1-t)\frac{d}{dt}\left(P_l^{(0,0)}(1-2t)\right) = 1.$$

Let us take

$$D(s) = -\overline{\alpha}\beta q_2\left(|\alpha|^2\right)E_- + (s+1)q_1\left(|\alpha|^2\right)I.$$

If we set $t = |\alpha|^2$ (recall that $|\alpha|^2 + |\beta|^2 = 1$), we have $(1-t) = |\beta|^2$; then by Lemma 2.2,

$$D(s)\left(t^l_{0,0}(g)\right)^{(s+1)} = (s+1)\left(t^l_{0,0}(g)\right)^s,$$

and so we have proved Theorem 2.1 in this particular case. The underlying idea here is the same as that used in [1].

To complete the proof of Theorem 2.3, we shall need some lemmas. Hereafter, we set $P := P^l_{m,n}$, $Q := Q^l_{m,n}$, $R := R^l_{m,n}$, and $S := S^l_{m,n}$.

LEMMA 2.5. (1) *If $m+n \geq 0$ and $m-n \geq 0$, there exist differential operators D_α, D_β, and D_P such that*

$$D_\alpha\left(t^l_{m,n}(g)\right)^{s+1} = p_\alpha(s)\left(t^l_{m,n}(g)\right)^s \overline{\beta}^{2m}\left(P\left(|\alpha|^2\right)\right)^{m+n+1}$$

$$D_\beta\left(t^l_{m,n}(g)\right)^{s+1} = p_\beta(s)\left(t^l_{m,n}(g)\right)^s \overline{\alpha}^{2m}\left(P\left(|\alpha|^2\right)\right)^{m-n+1}$$

$$D_P\left(t^l_{m,n}(g)\right)^{s+1} = (s+1)\left(t^l_{m,n}(g)\right)^s \overline{\alpha}^{m+n+1}\overline{\beta}^{m-n+1}\alpha P'\left(|\alpha|^2\right).$$

(2) *If* $m+n<0$ *and* $m-n<0$, *there exist differential operators* D_α, D_β, *and* D_S *such that*

$$D_\alpha\left(t^l_{m,n}(g)\right)^{s+1} = p_\alpha(s)\left(t^l_{m,n}(g)\right)^s \beta^{-2m}\left(S(|\alpha|^2)\right)^{-(m+n)+1}$$

$$D_\beta\left(t^l_{m,n}(g)\right)^{s+1} = p_\beta(s)\left(t^l_{m,n}(g)\right)^s \alpha^{-2m}\left(S(|\alpha|^2)\right)^{-(m-n)+1}$$

$$D_S\left(t^l_{m,n}(g)\right)^{s+1} = (s+1)\left(t^l_{m,n}(g)\right)^s \alpha^{-(m+n)+1}\beta^{-(m-n)+1}\overline{\beta}S'(|\alpha|^2),$$

where $p_\alpha(s)$, $p_\beta(s)$ *are the polynomials in s defined in* (2.1) *and* (2.2), *and* P' (*resp.* S') *is the derivative of* P (*resp.* S) *with respect to* t.

Proof. Let us assume that $m+n\geq 0$ and $m-n\geq 0$ (excluding the case $m+n=0$ and $m-n=0$). We will start by constructing D_α. Let D_1 be the following differential operator:

$$D_1 = -P(|\alpha|^2)E_- - (s+1)\alpha\overline{\beta}P'(|\alpha|^2)I.$$

Then by Example 2.4 and the Chain Rule, we have

$$D_1\left(t^l_{m,n}(g)\right)^{s+1} = (m+n)(s+1)\overline{\alpha}^{(m+n)(s+1)-1} \times \overline{\beta}^{(m-n)(s+1)+1}\left(P(|\alpha|^2)\right)^{(s+1)+1}.$$

Observe that if we consider our matrix coefficient as the product of the following factors:

$$\psi_\alpha = \overline{\alpha}^{(m+n)(s+1)} \quad \psi_\beta = \overline{\beta}^{(m-n)(s+1)} \quad \psi_P = \left(P(|\alpha|^2)\right)^{s+1},$$

we decrease in one the exponent of ψ_α, and increase in one the exponents of ψ_β and ψ_P. Take now

$$D_2 = -P(|\alpha|^2)E_- - (s+2)\alpha\overline{\beta}P'(|\alpha|^2)I.$$

Then

$$D_2\left(D_1\left(t^l_{m,n}(g)\right)^{s+1}\right) = (((m+n)(s+1))((m+n)(s+1)-1)) \times \overline{\alpha}^{(m+n)(s+1)-2}\overline{\beta}^{(m-n)(s+1)+2}\left(P(|\alpha|^2)\right)^{(s+1)+2}.$$

Let us define

$$D_j = -P(|\alpha|^2)E_- - (s+j)\alpha\overline{\beta}P'(|\alpha|^2)I$$

and take $D_\alpha = D_{(m+n)} \dots D_1$. Then it is easy to see that

$$D_\alpha\left(t^l_{m,n}(g)\right)^{s+1} = p_\alpha(s)\left(t^l_{m,n}(g)\right)^s \overline{\beta}^{2m}\left(P(|\alpha|^2)\right)^{1+m+n}.$$

Analogously we shall define D_β. Consider

$$T_1 = -P(|\alpha|^2)E_+ + (s+1)\overline{\alpha}\beta P'(|\alpha|^2)I.$$

Using Example 2.4, we have

$$T_1\left(t^l_{m,n}(g)\right)^{s+1} = -(m-n)(s+1)\overline{\alpha}^{(m+n)(s+1)+1} \times \overline{\beta}^{(m-n)(s+1)-1}\left(P(|\alpha|^2)\right)^{(s+1)+1}.$$

Observe that in this case, we decrease in one the exponent of ψ_β, and increase in one the exponents of ψ_α and ψ_P.

Let us define $T_j = -P(|\alpha|^2)E_+ + (s+j)\overline{\alpha}\beta P'(|\alpha|^2)I$ and

$$D_\beta = T_{(m-n)} \dots T_1.$$

Then

$$\begin{aligned} D_\beta\left(t^l_{m,n}(g)\right)^{s+1} &= \left[\prod_{j=0}^{m-n-1}((m-n)(s+1)-j)\right]\overline{\alpha}^{(m+n)(s+1)+m-n} \\ &\quad \times \overline{\beta}^{(m-n)s}\left(P(|\alpha|^2)\right)^{(s+1)+m-n} \\ &= p_\beta(s)\left(t^l_{m,n}(g)\right)^s \overline{\alpha}^{2m}\left(P(|\alpha|^2)\right)^{1+m-n}. \end{aligned}$$

Now we shall define D_P. In this case D_P is an order 1 differential operator, and it is defined as follows:

$$D_P = -\overline{\alpha}E_- - (m+n)(s+1)\overline{\beta}I.$$

Hence we have

$$D_P\left(t^l_{m,n}(g)\right)^{s+1} = (s+1)\left(t^l_{m,n}(g)\right)^s \overline{\alpha}^{(m+n+1)}\overline{\beta}^{m-n+1}\alpha P'(|\alpha|^2),$$

and so we have completed the proof of the first part of our result. The second is completely analogous if we consider

$$D_j = S(|\alpha|^2)E_+ - (s+j)\overline{\alpha}\beta S'(|\alpha|^2)I$$

$$T_j = S(|\alpha|^2)E_- + (s+j)\alpha\overline{\beta}S'(|\alpha|^2)I$$

and take

$$D_\alpha = D_{-(m+n)} \dots D_1, \quad D_\beta = T_{-(m-n)} \dots T_1$$

and

$$D_S = -\beta E_- + (-(m-n)(s+1))\,\alpha I.$$

∎

We have an analogous lemma for the remaining choices of m and n.

LEMMA 2.6. (1) *If $m+n \geq 0$ and $m-n < 0$, there exist differential operators D_α, D_β, and D_R such that*

$$D_\alpha\big(t^l_{m,n}(g)\big)^{s+1} = p_\alpha(s)\big(t^l_{m,n}(g)\big)^s \overline{\beta}^{m+n}\beta^{2n}\big(R(|\alpha|^2)\big)^{m+n+1}$$

$$D_\beta\big(t^l_{m,n}(g)\big)^{s+1} = p_\beta(s)\big(t^l_{m,n}(g)\big)^s \alpha^{-(m-n)}\,\overline{\alpha}^{2n}\big(R(|\alpha|^2)\big)^{-(m-n)+1}$$

$$D_R\big(t^l_{m,n}(g)\big)^{s+1} = (s+1)\big(t^l_{m,n}(g)\big)^s \overline{\alpha}^{m+n+1}\beta^{-(m-n)+1}\alpha\overline{\beta}R'\big(|\alpha|^2\big).$$

(2) *If $m+n < 0$ and $m-n \geq 0$, there exist differential operators D_α, D_β, and D_Q such that*

$$D_\alpha\big(t^l_{m,n}(g)\big)^{s+1} = p_\alpha(s)\big(t^l_{m,n}(g)\big)^s \beta^{-(m+n)}\overline{\beta}^{-2n}\big(Q(|\alpha|^2)\big)^{-(m+n)+1}$$

$$D_\beta\big(t^l_{m,n}(g)\big)^{s+1} = p_\beta(s)\big(t^l_{m,n}(g)\big)^s \overline{\alpha}^{(m-n)}\alpha^{-2n}\big(Q(|\alpha|^2)\big)^{(m-n)+1}$$

$$D_Q\big(t^l_{m,n}(g)\big)^{s+1} = (s+1)\big(t^l_{m,n}(g)\big)^s \alpha^{-(m+n)+1}\overline{\beta}^{(m-n)+1}\,\overline{\alpha}\beta Q'\big(|\alpha|^2\big),$$

where $p_\alpha(s)$, $p_\beta(s)$ are the polynomials defined in (2.1) *and* (2.2), *and R'* (*resp. Q'*) *is the derivative of R* (*resp. Q*) *in t.*

Proof. Let us assume that $m+n \geq 0$ and $m-n < 0$. In this case

$$D_\alpha = D_{(m+n)} \dots D_1,$$

where

$$D_j = -\beta R(|\alpha|^2)E_- + (-(m-n)(s+1) + j - 1)\,\alpha R(|\alpha|^2)I$$
$$- \beta(s+j)\,\alpha\overline{\beta}R'(|\alpha|^2)I.$$

Then by Example 2.4 and the Chain Rule, we have

$$D_\alpha\big(t^l_{m,n}(g)\big)^{s+1} = \left[\prod_{j=0}^{m+n-1}((m+n)(s+1)-j)\right]\overline{\alpha}^{(m+n)s}$$

$$\times\, \beta^{(-(m-n)(s+1))+m+n}\overline{\beta}^{m+n}\big(R(|\alpha|^2)\big)^{(s+1)+m+n}$$

$$= p_\alpha(s)\big(t^l_{m,n}(g)\big)^s\beta^{2n}\overline{\beta}^{m+n}\big(R(|\alpha|^2)\big)^{m+n+1},$$

where p_α is the polynomial defined in (2.1).

Analogously, we shall define D_β. Let us take

$$T_j = \overline{\alpha}R(|\alpha|^2)E_- + ((m+n)(s+1)+j-1)\,\overline{\beta}R(|\alpha|^2)I$$

$$+(s+j)\,\overline{\alpha}\alpha\overline{\beta}R'(|\alpha|^2)I$$

and $D_\beta = T_{-(m-n)}\dots T_1$. Hence we have

$$D_\beta\big(t^l_{m,n}(g)\big)^{s+1} = \left[\prod_{j=0}^{-(m-n)-1}(-(m-n)(s+1)-j)\right]$$

$$\times\, \overline{\alpha}^{(m+n)(s+1)+(-(m-n))}$$

$$\times\, \beta^{-(m-n)s}\big(R(|\alpha|^2)\big)^{(s+1)+(-(m-n))}$$

$$= p_\beta(s)\big(t^l_{m,n}(g)\big)^s\overline{\alpha}^{2n}\alpha^{-(m-n)}\big(R(|\alpha|^2)\big)^{(-(m-n)+1)}.$$

Finally, we shall define D_R as follows:

$$D_R = -\overline{\alpha}\beta E_- - (m+n)(s+1)|\beta|^2 I + (-(m-n)(s+1))|\alpha|^2 I.$$

And we have

$$D_R\big(t^l_{m,n}(g)\big)^{s+1} = (s+1)\big(t^l_{m,n}(g)\big)^s\overline{\alpha}^{(m+n+1)}\beta^{(-(m-n)+1)}\alpha\overline{\beta}R'(|\alpha|^2).$$

So we have completed the proof of (1). Part (2) is completely analogous. We just consider

$$D_j = Q(|\alpha|^2)\overline{\beta}E_+ + ((m-n)(s+1)+j-1)\,\overline{\alpha}Q(|\alpha|^2)I$$

$$-(s+j)|\beta|^2\,\overline{\alpha}Q'(|\alpha|^2)I$$

$$T_j = -Q(|\alpha|^2)\alpha E_+ + ((-(m+n))(s+1)+j-1)\,\beta Q(|\alpha|^2)I$$

$$+(s+j)|\alpha|^2\beta Q'(|\alpha|^2)I$$

and take

$$D_\alpha = D_{-(m+n)} \dots D_1, \quad D_\beta = T_{(m-n)} \dots T_1$$

and

$$D_Q = \alpha \overline{\beta} E_+ - \big((-(m+n))(s+1)|\beta|^2 + (m-n)(s+1)|\alpha|^2 \big) I.$$

∎

Remark. Observe that the exponents of α, β, $\overline{\alpha}$, and $\overline{\beta}$ in Lemmas 2.5 and 2.6 are always nonnegative.

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. The case $m+n=0$ and $m-n=0$ has already been demonstrated. We shall prove the theorem just for $m+n \geq 0$ and $m-n \geq 0$, excluding the case $m+n=0$ and $m-n=0$.

The remaining cases are completely analogous for Lemmas 2.5 and 2.6.

Observe that

$$\beta^{2m} D_\alpha \big(t^l_{m,n}(g) \big)^{s+1} = p_\alpha(s) \big(t^l_{m,n}(g) \big)^s r_\alpha$$

$$\alpha^{2m} D_\beta \big(t^l_{m,n}(g) \big)^{s+1} = p_\beta(s) \big(t^l_{m,n}(g) \big)^s r_\beta$$

$$\alpha^{(m+n+1)} \beta^{m-n+1}\, \overline{\alpha} D_P \big(t^l_{m,n}(g) \big)^{s+1} = (s+1) \big(t^l_{m,n}(g) \big)^s r_P,$$

where

$$r_\alpha = \big(|\beta|^2 \big)^{2m} \big(P(|\alpha|^2) \big)^{1+m+n}$$

$$r_\beta = \big(|\alpha|^2 \big)^{2m} \big(P(|\alpha|^2) \big)^{1+m-n}$$

$$r_P = \big(|\alpha|^2 \big)^{(m+n+2)} \big(|\beta|^2 \big)^{m-n+1} P'\big(|\alpha|^2\big).$$

If we set $t = |\alpha|^2$, we have $(1-t) = |\beta|^2$, and we obtain the following polynomials:

$$r_\alpha(t) = (1-t)^{2m} (P(t))^{1+m+n}$$

$$r_\beta(t) = t^{2m} (P(t))^{1+m-n}$$

$$r_P(t) = t^{(m+n+2)} (1-t)^{m-n+1} P'(t).$$

By Lemma 2.2, it is easy to see that these polynomials are coprime, i.e., there exist q_α, q_β, and q_P in $\mathbb{R}[t]$ such that

$$r_\alpha(t) q_\alpha(t) + r_\beta(t) q_\beta(t) + r_P(t) q_P(t) = 1 \qquad \text{for all } t \in R. \tag{2.6}$$

Now we shall define the operator $D(s)$. Let $p(s)$ be the least common multiple of p_α and p_β. Observe that $s+1$ divides $p(s)$. Define

$$\begin{aligned} D(s) = {} & \frac{p(s)}{p_\alpha(s)} \beta^{2m} q_\alpha(|\alpha|^2) D_\alpha + \frac{p(s)}{p_\beta(s)} \alpha^{2m} q_\beta(|\alpha|^2) D_\beta \\ & + \frac{p(s)}{s+1} \alpha^{m+n+1} \beta^{m-n} \overline{\beta} q_P(|\alpha|^2) D_P . \end{aligned} \tag{2.7}$$

Then by (2.6) we have

$$\begin{aligned} & D(s)\left(t^l_{m,n}(g)\right)^{s+1} \\ & \quad = p(s)\left(t^l_{m,n}(g)\right)^s \left[r_\alpha(|\alpha|^2) q_\alpha(|\alpha|^2) + r_\beta(|\alpha|^2) q_\beta(|\alpha|^2) \right. \\ & \qquad \left. + r_P(|\alpha|^2) q_P(|\alpha|^2) \right] = p(s)\left(t^l_{m,n}(g)\right)^s , \end{aligned}$$

and so we have proved the desired result. ∎

3. THE BERNSTEIN POLYNOMIAL

Let $G = SU(2)$. Recall that the polynomial that satisfies Eq. (1.2) is not unique. It is easy to see that the set of polynomials that satisfies this functional equation form an ideal, and since $\mathbb{C}[s]$ is a principal ideal domain, we can consider the monic generator of this ideal, and this is known as the *Bernstein polynomial.*

Fix again m, n, and l. Let us define $b(s) = (s+1)$ if $m+n=0$ and $m-n=0$, and otherwise let $b(s)$ be the least common multiple of

$$b_\alpha(s) = \begin{cases} \left(\prod_{j=0}^{|m+n|-1} \left((s+1) - \dfrac{j}{(|m+n|)} \right) \right) & \text{if } m+n \neq 0 \\ 1 & \text{if } m+n = 0 \end{cases}$$

and

$$b_\beta(s) = \begin{cases} \left(\prod_{j=0}^{|m-n|-1} \left((s+1) - \dfrac{j}{(|m-n|)} \right) \right) & \text{if } m-n \neq 0 \\ 1 & \text{if } m-n = 0 \end{cases}$$

(cf. (2.1) and (2.2)). In the following theorem, we shall prove that $b(s)$ is the Bernstein polynomial attached to $t^l_{m,n}(g)$.

THEOREM 3.1. *Let $T(s)$ be a differential operator with coefficients in $A(G)[s]$ that satisfies*

$$T(s)\left(t^l_{m,n}(g)\right)^{s+1} = d(s)\left(t^l_{m,n}(g)\right)^{s},$$

$d(s) \in \mathbb{C}[s]$. *Then $b(s)$ divides $d(s)$.*

Proof. Let $I = (i, j, k)$ be a 3-tuple with $i, j, k \in \mathbb{Z}_{\geq 0}$ and $D^I = E^i_+ E^j_0 E^k_-$. Then for each 3-tuple $I \neq 0$, we have

$$D^I\left(t^l_{m,n}(g)\right)^{s+1} = h_I(g,s)\left(t^l_{m,n}(g)\right)^{s}, \tag{3.1}$$

where $h_I(g,s)$ is a rational function in α, β, $\overline{\alpha}$, and $\overline{\beta}$.

By hypothesis, $T(s)$ is a differential operator with coefficients in $A(G)[s]$, so it can be written as

$$T(s) = \sum_I f_I(g,s) D^I, \quad f_I(g,s) \in A(G)[s].$$

By hypothesis and (3.1), we have that

$$\sum_I f_I(g,s) h_I(g,s) = d(s). \tag{3.2}$$

To see that $b(s)$ divides $d(s)$, it is enough to see that $b_\alpha(s)$, $b_\beta(s)$, and $s+1$ divide $d(s)$.

Let us assume that $m+n \geq 0$ and $m-n \geq 0$ (exclude the case $m+n = 0$ and $m-n = 0$; this will be proved later). As usual, we only prove the theorem in this case because the other cases are completely analogous. By (1) in Lemma 2.2 we have

$$\left(t^l_{m,n}(g)\right)^{s+1} = \overline{\alpha}^{(m+n)(s+1)} w^{s+1}_\alpha(g), \tag{3.3}$$

where $w_\alpha(g) = \overline{\beta}^{(m-n)}(P(|\alpha|^2))$.

We shall show that b_α divides $d(s)$. By the product rule and Example 2.4 we have

$$\begin{aligned}
&D^I\big(t^l_{m,n}(g)\big)^{s+1}\\
&\quad = E^i_+ E^j_0 E^k_-\big(\overline{\alpha}^{(m+n)(s+1)} w_\alpha^{s+1}(g)\big)\\
&\quad = E^i_+\Bigg(\sum_{u=0}^{j}\sum_{v=0}^{k}\binom{j}{u}\binom{k}{v} E^u_0 E^v_-\big(\overline{\alpha}^{(m+n)(s+1)}\big)\\
&\qquad\qquad E_0^{j-u}E_-^{k-v}\big(w_\alpha^{s+1}(g)\big)\Bigg)\\
&\quad = E^i_+\Bigg(\sum_{u=0}^{j}\sum_{v=0}^{k}\binom{j}{u}\binom{k}{v} A_{u,v}(s)\prod_{l=0}^{v-1}((m+n)(s+1)-l)\\
&\qquad\qquad \overline{\alpha}^{(m+n)(s+1)-v}\big(-\overline{\beta}\big)^v E_0^{j-u}E_-^{k-v}\big(w_\alpha^{s+1}(g)\big)\Bigg)\\
&\quad = \big(t^l_{m,n}(g)\big)^s \sum_{u=0}^{j}\sum_{v=0}^{k}\binom{j}{u}\binom{k}{v} A_{u,v}(s)\prod_{l=0}^{v-1}((m+n)(s+1)-l)\\
&\qquad \overline{\alpha}^{(m+n)-v}E^i_+\left(\big(-\overline{\beta}\big)^v \frac{E_0^{j-u}E_-^{k-v}\big(w_\alpha^{s+1}(g)\big)}{w_\alpha^s(g)}\right),
\end{aligned}$$

where $A_{u,v}(s)$ is the polynomial given by the action of E^u_0 on $E^v_-(\overline{\alpha}^{(m+n)(s+1)})$. Then by (3.1),

$$\begin{aligned}
h_I(g,s) = \sum_{u=0}^{j}\sum_{v=0}^{k}\binom{j}{u}\binom{k}{v} A_{u,v}(s)\prod_{l=0}^{v-1}((m+n)(s+1)-l)\\
\overline{\alpha}^{(m+n)-v}E^i_+\left(\big(-\overline{\beta}\big)^v \frac{E_0^{j-u}E_-^{k-v}\big(w_\alpha^{s+1}(g)\big)}{w_\alpha^s(g)}\right).
\end{aligned}$$

Let $\delta_I \in \mathbb{Z}_{\geq 0}$ the greatest nonnegative integer such that

$$\frac{f_I(g,s)}{\overline{\alpha}^{\delta_I}} \in A(G)[s].$$

Let us take

$$J = \{v \geq 0 \colon \delta_I + m + n - v > 0\}$$

$$K = \{v \geq 0 \colon \delta_I + m + n - v < 0\}$$

and

$$L = \{v \geq 0 \colon \delta_I + m + n - v = 0\}.$$

Hence if we define

$$A_{+,I}(g,s) = \sum_{v \in J} \sum_{u=0}^{j} \binom{j}{u}\binom{k}{v} A_{u,v}(s) \prod_{l=0}^{v-1} ((m+n)(s+1) - l)$$
$$f_I(g,s)\,\overline{\alpha}^{(m+n)-v} E_+^i \left((-\overline{\beta})^v \frac{E_0^{j-u} E_-^{k-v}\left(w_\alpha^{s+1}(g)\right)}{w_\alpha^s(g)} \right)$$

$$A_{-,I}(g,s) = \sum_{v \in K} \sum_{u=0}^{j} \binom{j}{u}\binom{k}{v} A_{u,v}(s) \prod_{l=0}^{v-1} ((m+n)(s+1) - l)$$
$$f_I(g,s)\,\overline{\alpha}^{(m+n)-v} E_+^i \left((-\overline{\beta})^v \frac{E_0^{j-u} E_-^{k-v}\left(w_\alpha^{s+1}(g)\right)}{w_\alpha^s(g)} \right)$$

and

$$A_{\delta_I}(g,s) = \sum_{v \in L} \sum_{u=0}^{j} \binom{j}{u}\binom{k}{v} A_{u,v}(s) f_I(g,s)$$
$$\overline{\alpha}^{(m+n)-v} E_+^i \left((-\overline{\beta})^v \frac{E_0^{j-u} E_-^{k-v}\left(w_\alpha^{s+1}(g)\right)}{w_\alpha^s(g)} \right),$$

we have

$$f_I(g,s) h_I(g,s) = A_{+,I}(g,s) + A_{-,I}(g,s)$$
$$+ \left(\prod_{l=0}^{\delta_I + m + n - 1} ((m+n)(s+1) - l) \right) A_{\delta_I}(g,s).$$

By (3.1),

$$d(s) = \sum_I f_I(g,s) h_I(g,s)$$
$$= \sum_I A_{+,I}(g,s) + \sum_I A_{-,I}(g,s)$$
$$+ \sum_I \prod_{l=0}^{\delta_I + m + n - 1} ((m+n)(s+1) - l)\, A_{\delta_I}(g,s).$$

By definition of the set K, we have that $A_{-,I}(g,s)$ has a pole in $\overline{\alpha}$, but $d(s)$ has not, then $\sum_I A_{-,I}(g,s) = 0$. Hence

$$d(s) = \sum_I A_{+,I}(g,s) + \sum_I \prod_{l=0}^{\delta_I+m+n-1} ((m+n)(s+1)-l)A_{\delta_I}(g,s).$$

Evaluating both sides of this equation in

$$g_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.4}$$

we have

$$d(s) = \sum_I \prod_{l=0}^{\delta_I+m+n-1} ((m+n)(s+1)-l)A_{\delta_I}(g_0,s).$$

Since $b_\alpha(s)$ divides $(\prod_{l=0}^{\delta_I+m+n-1}((m+n)(s+1)-l))$ for all I, we deduce that $b_\alpha(s)$ divides $d(s)$.

In the same way we can see that b_β divides $d(s)$.

It remains to handle the case $m+n=0$ and $m-n=0$. In this case, by Lemma 2.2, the matrix coefficient is $t^l_{0,0}(g) = P(|\alpha|^2)$, where P is related with a Legendre polynomial. Let us see, briefly, that $s+1$ divides $d(s)$. It is easy to see that for each n-tuple $I \neq 0$, we have

$$D^I\big(t^l_{0,0}(g)\big)^{s+1} = (s+1)\tilde{h}_I(g,s)\big(t^l_{0,0}(g)\big)^s, \tag{3.5}$$

where $\tilde{h}_I(g,s)$ is a rational function in α, β, $\overline{\alpha}$, and $\overline{\beta}$ and a polynomial in s. As in the case above, we can see

$$(s+1)\sum_I f_I(g,s)\tilde{h}_I(g,s) + f_{(0,0,0)}(g,s)P(|\alpha|^2)d(s). \tag{3.6}$$

By (3.2) and Lemma 2.2,

$$\big(t^l_{m,n}(g)\big)^{s+1} = \big(P(|\alpha|^2)\big)^{s+1} = (|\alpha|^2 - a)^{s+1}q(g), \tag{3.7}$$

where a is a simple root of P belonging to $(0,1)$ and $q(g) = C_{m,n,l}((P(|\alpha|^2))^{s+1}/(|\alpha|^2-a)^{s+1})$.

We can show again that

$$f_I(g,s)h_I(g,s) = B_{+,I}(g,s) + B_{-,I}(g,s) + (s+1)B_{\gamma_I}(g,s)$$

for each I, were γ_I is the greatest nonnegative integer such that $f_I(g,s)/(|\alpha|^2-a)^{\gamma_I} \in A(G)[s]$ and $B_{-,I}(g,s)$ has a pole in $(|\alpha|^2-a)$. Adding both sides over I, we have

$$\begin{aligned} d(s) &= \sum_I f_I(g,s)\tilde{h}_I(g,s) \\ &= \sum_I B_{+,I}(g,s) + \sum_I B_{-,I}(g,s) \\ &\quad + \sum_I (s+1)B_{\gamma_I}(g,s). \end{aligned}$$

Observe that since $d(s)$ has no pole, $\sum_I B_{-,I}(g,s) = 0$. Hence,

$$d(s) = \sum_I B_{+,I}(g,s) + \sum_I (s+1)B_{\gamma_I}(g,s).$$

Evaluating both sides of the equation in

$$g_0 = \begin{pmatrix} \sqrt{a} & \sqrt{1-a} \\ -\sqrt{1-a} & \sqrt{a} \end{pmatrix},$$

we have proved our theorem. ∎

4. ANALYTIC CONTINUATION OF DISTRIBUTIONS

Recall that $G = SU(2)$. Let us consider $h^l_{m,n}(g) = |t^l_{m,n}(g)|^2$. Using the same techniques developed in Section 2, we can prove the analogs of Theorem 2.1 for this function. More precisely,

THEOREM 4.1. *Let $h^l_{m,n}(g)$ be as defined above. Then there exists a differential operator $D(s)$ with coefficients in $A(G)[s]$ such that*

$$D(s)\left(h^l_{m,n}(g)\right)^{s+1} = p(s)\left(h^l_{m,n}(g)\right)^s, \tag{4.1}$$

where $p(s) \in \mathbb{C}[s]$ is the least common multiple of $(p_\alpha(s))^2$, $(p_\beta(s))^2$, and $2(s+1)(2(s+1)-1)$, *where p_α and p_β were defined in* (2.1) *and* (2.2).

Now we can take $(h^l_{m,n}(g))^s$, where $s \in \mathbb{C}$, and this is a continuous function in G. Let us consider the following function on s:

$$\Gamma_f(s) = \int_{SU(2)} \left(h^l_{m,n}(g)\right)^s f(g)\, dg,$$

where $f \in C(G)$ and dg is the Haar measure on G. It easy to see that

$\Gamma_f(s)$ is an analytic function for $\mathrm{Re}(s) \geq 0$. Theorem 4.1 allows us to give meromorphic continuation to $\mathbb{C}$ of Γ_f, since

$$\begin{aligned}
\Gamma_f(s) &= \int_{SU(2)} \left(h^l_{m,n}(g)\right)^s f(g)\,dg \\
&= \frac{1}{p(s)} \int_{SU(2)} D(s)\left(h^l_{m,n}(g)\right)^{s+1} f(g)\,dg \\
&= \frac{1}{p(s)} \int_{SU(2)} \left(h^l_{m,n}(g)\right)^{s+1} D^*(s) f(g)\,dg \\
&= \frac{1}{p(s)} \Gamma_{D^*(s)f}(s+1),
\end{aligned}$$

where $D^*(s)$ is the adjoint of $D(s)$. Hence, iterating this formula, we obtain the desired meromorphic continuation, and its poles are related to the zeros of the polynomial $p(s)$.

5. GENERAL CASE

Reassembling the proof of Theorem 2.1 in [6] by Kashiwara, we can prove the following theorem, which gives us the description of the Bernstein polynomial for $B_- \times B$-semi-invariant functions on a semisimple algebraic Lie group G. This idea is due mostly to J. Vargas. Let us introduce some notation.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b}$, and let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{b}$.

Let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$. For each $\alpha \in \Delta$, let h_α be the coroot of α. Let Δ^+ be the set of positive roots given by $\mathfrak{b}$, and let ρ be the half-sum of positive roots.

Set $(x, m) = x(x+1)(x+2)\cdots(x+m-1)$.

Let G be a simply connected algebraic group with Lie algebra $\mathfrak{g}$, and let B, T, and N be the subgroups of G with Lie algebras $\mathfrak{b}$, $\mathfrak{h}$, and $\mathfrak{n}$, respectively. Let B_- be the opposite Borel subgroup.

DEFINITION 5.1. A $B_- \times B$-semi-invariant function f in G is a regular function in G that satisfies

$$f(b'gb) = \chi'(b')\chi(b)f(g)$$

for $b' \in B_-$, $g \in G$, $b \in B$, where χ' and χ are characters of B_- and B, respectively.

Remark 5.2. The semigroup of $B_{-}\times B$-semi-invariant functions in G is parameterized by the set $P_{+}=\{\mu\in\mathfrak{h}^{*}: h_{\alpha}(\mu)\in\mathbb{Z}_{\geq 0}$ for all $\alpha\in\Delta^{+}\}$ of dominant integral weights. More precisely, for $\mu\in P_{+}$, take V_{μ} the irreducible representation of G with highest weight μ, v_{μ} a highest weight vector of V_{μ}, and $v_{-\mu}$ a lowest weight vector of V_{μ}^{*}. We normalize them such that $\langle v_{\mu}, v_{-\mu}\rangle = 1$. Then the regular function

$$f_{\mu}(g) = \langle g v_{\mu}, v_{-\mu}\rangle$$

is a $B_{-}\times B$-semi-invariant, and any $B_{-}\times B$-semi-invariant function is a constant multiple of some f_{μ}.

So now we can state the following theorem, which gives us the Bernstein polynomial attached to f_{μ}, with $\mu\in \mathrm{P}_{+}$.

THEOREM 5.3. *For each dominant integral weight μ, there exists a differential operator P_{μ} such that*

$$P_{\mu} f_{\mu}^{(s+1)} = b(s) f_{\mu}^{s},$$

where $b(s) = \prod_{\alpha\in\Delta^{+}}((s+1)h_{\alpha}(\mu) + h_{\alpha}(\rho-\mu), h_{\alpha}(\mu))$. *Moreover, $b(s)$ is the Bernstein polynomial attached to f_{μ}.*

Remark 5.4. Observe that in [6], Kashiwara is shifting the parameter within the dominant integral weights. To reassemble the proof of Theorem 2.1 in [6], for our case, we should note that the reduction steps are almost the same, recalling that s is just a formal variable, and using the formal action defined in (1.1) for the last calculation.

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